COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 8 (2) (1988) 177—184

MATRICES WITH PRESCRIBED ROW, COLUMN AND BLOCK SUMS

Zh. A. CHERNYAK and A. A. CHERNYAK

Received 1 February 1985

Revised 8 September 1986

The main result of the paper is Theorem 1. It concerns the sets of integral symmetric matrices with given block partition and prescribed row, column and block sums. It is shown that by interchanges preserving these sums we can pass from any two matrices, one from each set, to the other two ones falling "close" together as much as possible. One of the direct corollaries of Theorem 1 is substantiating the fact that any realization of r-graphical integer-pair sequence can be obtained from any other one by r-switchings preserving edge degrees. This result is also of interest in connection with the problem of determining s-complete properties. In the special cases Theorem 1 includes a number of well-known results, some of which are presented.

The paper has been inspired by the researches concerned with the properties of graphical integer-pair sequences (edge degree sequences) and their realizations. These sequences extend the concepts and results of the well-known theory of degree sequences (i.e. vertex degree sequences) and described the structure of graphs much more than degree sequences [6], [7], [10], [15]. In view of this there arises the necessity of receiving effective auxiliary means for solving the problems dealing with integerpair sequences and adjacency matrices of their realizations. Note that in the degree sequence theory (as in the theory of integral matrices with prescribed row and column sums) such means are various theorems concerning getting any realization (matrix respectively) from any other by simple operations, such as switchings (interchanges), preserving vertex degrees (row and column sums respectively) [2], [3], [5], [11], [12].

The main result of our paper is Theorem 1. It concerns the sets of integral symmetric matrices with given block partition and prescribed row, column and block sums. It is shown that by interchanges preserving these sums, with some additional assumptions none of which can be ommited, we can pass from any two matrices (one from each set) to the other two ones falling "close" together as much as possible. One of the direct corollaries of Theorem 1 is substantiating the fact that any realization of r-graphical integer-pair sequence can be obtained from any other one by r-switchings, called r-exchanges, preserving edge degrees. This result is also of interest in connection with the problem of determining s-complete properties [4], [9]. In the special case of trivial block partition Theorem 1 includes a number of well-known results, in particular Changphaisan's Interchange Theorem and Kleitman—Wang—Kundu's K-factor Theorem, some of which are presented in the last section.

AMS subject classifications (1980): 05 B 20, 05 C 99

1. Terminology, notation

$A[I, J]$ denotes submatrix of A whose rows are indexed by I and whose columns are indexed by J. Let $\|A\|=\sum_{i,j}|a_{ij}|$, where a_{ij} are entries of matrix A. If $R=(r_1, \dots, r_m)$, $S=(s_1, \dots, s_n)$ are nonnegative integral vectors, $I=\{1, \dots, m\}=\bigcup_1^x I_i$, $J=\{1, \dots, n\}=\bigcup_1^z J_j$ are partitions, $K=[k_{ij}]$ is $x\times z$ integral matrix, $1\leqq r\leqq\infty$ then $U_1=U\left(R, S, K, r \mid \bigcup_1^x I_i, \bigcup_1^z J_j\right)$ denotes the set of all $m\times n$ integral matrices $A=[a_{ij}]$ satisfying the following conditions:

$$0 \leqq a_{ij} \leqq r, \quad \|A[i, J]\| = r_i, \quad \|A[I, j]\| = s_j,$$
$$\|A[I_i, J_j]\| = k_{ij} \quad \text{(block sums)},$$
$$\text{for} \quad i = 1, \dots, m, \quad j = 1, \dots, n.$$

If $r=1$ and partitions of I, J are trivial, i.e. $x=z=1$, we have class $U(R, S, 1)$ of (0, 1)-matrices [3].

Let $\bar{R}=(\bar{r}_1, \dots, \bar{r}_m)$, $\bar{S}=(\bar{s}_1, \dots, \bar{s}_n)$, $\bar{K}=[\bar{k}_{ij}]$. We say that sets U_1 and $U_2=U\left(\bar{R}, \bar{S}, \bar{K}, r \middle| \bigcup_1^x I_i, \bigcup_1^z J_j\right)$ are compatible on rows (on column respectively) if

$$|(r_i-\bar{r}_i)-(r_j-\bar{r}_j)| \leqq 1 \quad \left(|(s_i-\bar{s}_i)-(s_j-\bar{s}_j)| \leqq 1 \quad \text{respectively}\right)$$

whenever $i, j\in I_k$ ($i, j\in J_k$ respectively). We call sets of matrices compatible if they are both row and column compatible.

Denote by $U_3=U\left(R, K, r \middle| \bigcup_1^x I_i\right)$ the set of all $m\times m$ integral symmetric matrices $A=[a_{ij}]$ satisfying the following conditions:

$$0 \leqq a_{ij} \leqq r, \quad \text{trace} \quad (A) = 0, \quad \|A[i, I]\| = r_i, \quad \|A[I_i, I_j]\| = k_{ij}$$

for $i=1, \dots, m$, $j=1, \dots, m$. If $r=1$ and partition of I is trivial, i.e. $x=1$, we have class $U(R, 1)$ of (0, 1)-matrices. As seen below, the introduction of block sums k_{ij} is motivated by edge degree sequences. It may be remarked that if $|r_i-r_j|=1$, $|s_p-s_q|=1$ whenever $i, j\in I_k$, $p, q\in J_t$ then criterions for the nonemptiness of U_1, U_3 can be easily derived from constructive proof of Existens Theorem in [7].

Let $A=[a_{ij}]$ be an integral matrix such that $a_{kl}>0$, $a_{gh}>0$, $a_{kh}<r$, $a_{gl}<r$. We define a r-interchange $f=(kglh)$ to be a transformation which replaces the 2×2 submatrix

$$\begin{bmatrix} a_{kl} & a_{kh} \\ a_{gl} & a_{gh} \end{bmatrix}$$

of matrix A by 2×2 submatrix

$$\begin{bmatrix} a_{kl}-1 & a_{kh}+1 \\ a_{gl}+1 & a_{gh}-1 \end{bmatrix}.$$

We define a r-transfer $f=(kglh)$ to be a r-interchange in which either $\{k, g\} \subseteq I_s$, or $\{l, h\} \subseteq J_s$ for some s. A symmetric r-interchange (or symmetric r-transfer) additionaly replaces the 2×2 submatrix

$$\begin{bmatrix} a_{lk} & a_{lg} \\ a_{hk} & a_{hg} \end{bmatrix}$$

of symmetric matrix A by 2×2 submatrix

$$\begin{bmatrix} a_{lk}-1 & a_{lg}+1 \\ a_{hk}+1 & a_{hg}-1 \end{bmatrix}$$

with restriction that $\{l, h\} \cap \{k, g\} = \emptyset$. Obviously, r_i, s_j, k_{ij} are invariant under r-transfers.

By r-graph is meant a loopless undirected graph in which no two vertices are joined by more than r edges [1]. 1-graphs and ∞-graphs are correspondently graphs and multigraphs. R-switching $t=(abcd)$ where a, b, c, d are distinct is a transformation of a r-graph G to a r-graph H by replacing two edges ac, bd in G by two edges ad, bc [5]. By r-exchange is meant a r-switching $t=(abcd)$ in which degrees of vertices a, b are equal [8].

A sequence

$$R = r_1, r_2, \ldots, r_m \tag{1}$$

is called r-graphical if there exists a r-graph G with m vertices $v_1, \ldots, v_m$ such that degree of v_i equals r_i for $i=1, \ldots, m$. Such a r-graph G is called a realization of R. By degree of edge is meant an unordered pair of degrees of its end vertices. Edge degree sequence of r-graph G without isolated vertices (r-graphical integer-pair sequence) can be written in the following form:

$$\{(d_i, d_j)^{k_{ij}}\} \quad 1 \leq i \leq j \leq x \tag{2}$$

where $d_1, \ldots, d_x$ is a degree set of r-graph G (realization respectively). It may be remarked that for r-graph G without isolated vertices edge degree sequence uniquely determines its vertex degree sequence.

The property P is called switching complete (s-complete) if for any two graphs G, H with the same degree sequence, both satisfying property P, there is a sequence of 1-switchings which transforms G into a graph isomorphic to G, all intermediate graphs satisfying property P. The problem of determining s-complete properties has been posed in [4].

2. The main result

Theorem 1. *Let* $A \in U_3$ *and* $B \in U_4 = U(\bar{R}, \bar{K}, r \mid \bigcup_1^x I_i)$. *For each* I_i *there exists a* d_i *with* $r_j - \bar{r}_j = d_i$ *or* $d_i + 1$ *for* $j \in I_i$. *If* $x > 1$ *then for each* I_i *there exists* e_i, g_i *with* $r_j = e_i$ *or* $e_i + 1$ *and* $\bar{r}_j = g_i$ *or* $g_i + 1$ *for* $j \in I_i$.

Then there exist sequences of r-transfers taking A, B into matrices M, N which differ by the least possible:

$$\|M-N\| = \sum_{s,t} |k_{st} - \bar{k}_{st}|,$$

Proof. Let $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]$, $C=A-B$. Suppose that $\|C\|>\sum|k_{st}-\bar{k}_{st}|$. Then there exist integers m, q such that

$$\|C[I_m, I_q]\| > |k_{mq}-\bar{k}_{mq}| = \Big|\sum_{\substack{i\in I_m\\ j\in J_q}} c_{ij}\Big|.$$

Hence $C[I_m, I_q]$ contains entries of opposite signs: $c_{ij}>0$, $c_{gh}<0$. It is assumed throughout this proof that r-transfers f, f' with odd (even respectively) subscripts are performed over matrix A (B respectively). Consider two cases.

Case 1. The sentence $(i=g)\vee(j=h)$ is true. Let, for example, $i=g$. Since $c_{jj}=c_{hh}=0$, $c_{jh}=c_{hj}$, $c_{ij}\geqq c_{ih}+2$ and

$$\Big|\sum_s c_{sj}-\sum_s c_{sh}\Big| = |(r_j-\bar{r}_j)-(r_h-\bar{r}_h)| \leqq 1$$

then there exist $k\neq j, h$ such that $c_{kj}<c_{kh}$. The last inequality implies $2r<a_{kh}+(r-a_{kj})+(r-b_{kh})+b_{kj}$.

In the right side of the above inequality all summands are nonnegative and not exceed r. Hence at least three of them are positive. It follows that there arise the following two possibilities:

$$\text{(3)}\qquad \begin{aligned} &1)\quad a_{kj}<r,\ a_{kh}>0;\\ &2)\quad b_{kj}>0,\ b_{kh}<r. \end{aligned}$$

Obviously, $a_{ij}>0$, $b_{ij}<r$, $a_{ih}<r$, $b_{ih}>0$. Let $f_1=(gkjh)$, $f_2=(ikhj)$. If possibility u) of (3) holds, where $u\in\{1, 2\}$, then the r-transfer f_u is allowable. Now let A_1, B_1 be matrices obtained from A, B respectively by performing the r-transfer f_u (one of A, B is not altered). Then since $c_{kh}>c_{kj}$ it follows that

$$\begin{aligned}(\|C\|-\|A_1-B_1\|)/2 &= |c_{ij}|+|c_{ih}|+|c_{kj}|+|c_{kh}|-|c_{ij}-1|-|c_{ih}+1|-|c_{kj}+1|-|c_{kh}-1| =\\ &= (a_{ij}-b_{ij})+(b_{ih}-a_{ih})-(a_{ij}-b_{ij}-1)-(b_{ih}-a_{ih}-1)+|c_{kj}|-|c_{kj}+1|+|c_{kh}|-\\ &\quad -|c_{kh}-1| = 2+|c_{kj}|-|c_{kj}+1|+|c_{kh}|-|c_{kh}-1| \geqq 2.\end{aligned}$$

Case 2. The sentence $(i=g)\vee(j=h)$ is false.

If $g=j$ then $c_{gi}=c_{ij}>0$, $(g, h), (g, i)\in I_m\times I_q$, $I_m=I_q$, i.e. the Case 1 takes place. The case $i=h$ is similar. A situation of Case 1 arises provided $|c_{ih}|+|c_{gj}|\neq 0$.

Suppose now $g\neq j$, $i\neq h$, $c_{ih}=c_{gj}=0$. As before $a_{ij}>0$, $b_{ij}<r$, $a_{gh}<r$, $b_{gh}>0$. Note that there exists $t\neq i, g$ such that $c_{tj}<c_{th}$. It follows that either $c_{tj}<0$ or $c_{th}>0$, hence provided $x=1$ a situation of Case 1 arises. So we can further suppose that $x>1$. Consider two subcases.

Subcase A. $|a_{gj}-a_{ih}|<r$. Then either $a_{ih}>0$, $a_{gj}>0$ or $a_{ih}<r$, $a_{gj}<r$. By analogy with the Case 1, taking in consideration $c_{ih}=c_{gj}=0$, we conclude that there are k, l such that $\{k, l\}\cap\{i, j, g, h\}=\emptyset$, $c_{kj}<c_{kh}$, $c_{il}<c_{gl}$, i.e. one of the possibilities of (3) holds and one of the following holds:

$$\text{(4)}\qquad \begin{aligned} &3)\quad a_{il}<r,\ a_{gl}>0;\\ &4)\quad b_{il}>0,\ b_{gl}<r. \end{aligned}$$

Denote $f_3=(igsl)$, $f_4=(gisl)$, $f_1'=(skjh)$, $f_2'=(skhj)$, $f_3'=(igjl)$, $f_4'=(gihl)$ Let the possibility u) of (3) and the possibility v) of (4) hold where $u\in\{1,2\}$, $v\in\{3,4\}$. If $a_{ih}>0$, $a_{gj}>0$ then we perform successively r-transfers f_u, f_v over corresponding matrices, in so doing substitute s by j in f_v provided $u=1$ and s by h in f_v provided $u=2$. If $a_{ih}<r$, $a_{gj}<r$ then we perform successively r-transfers f_v', f_u' over corresponding matrices, in so doing substitute s by g in f_u' provided $v=3$ and s by i in f_u' provided $v=4$. Now if A_1, B_1 are matrices obtained from A, B respectively by performing above r-transfers then taking into account inequalities

$$|c_{kj}|+|c_{kh}|-|c_{kj}+1|-|c_{kh}-1| \geqq 0,$$

$$|c_{il}|+|c_{gl}|-|c_{il}+1|-|c_{gl}-1| \geqq 0,$$

we have

$$(\|C\|-\|A_1-B_1\|)/2 \geqq |c_{ij}|+|c_{gh}|-|c_{ij}-1|-|c_{gh}+1| =$$

$$= c_{ij}-(c_{ij}-1)-c_{gh}+(c_{gh}+1) = 2.$$

Subcase B. $|a_{gj}-a_{ih}|=r$. Without loss of generality we assume that $a_{ih}=r$, $a_{gj}=0$. Let k, l be the same as in previous Subcase. If either $b_{kj}>b_{kh}$ or the possibility 2) of (3) holds then we perform the r-transfer f_2. Suppose that $b_{kj}\leqq b_{kh}$ and the possibility 1) of (3) holds. Since $b_{ij}<b_{ih}$, $b_{gj}<b_{gh}$ there exists $p\neq j, h, k, i, g$ such that $b_{pj}>b_{ph}$. Denote $f_5=(kphj)$, $f_6=(iphj)$. Now if $c_{pj}<c_{ph}$ then we perform the r-transfer f_6. If $c_{pj}\geqq c_{ph}$ then we perform successively r-transfers f_5, f_6 over corresponding matrices. In any of just listed possibilities, as a result of performing indicated r-transfers, b_{ij} increases by 1, b_{ih} decreases by 1 whereas the sum of absolute values of entries in positions (k,j), (k,h), (p,j), (p,h) of new matrix's difference does not increase in comparison with corresponding sum of matrix C. Next, similarly with k replaced by 1, we make a_{ih} be decreased by 1, a_{gh} be increased by 1 whereas the sum of absolute values of entries in rows i and g, other than in positions (i,h), (g,h), be not increased in comparison with corresponding sum of previous one. Thus, if A_1, B_1 are matrices obtained from A, B respectively by performing the above r-transfers then

$$(\|C\|-\|A_1-B_1\|)/2 \geqq |c_{ij}|-|c_{ij}-1|+|c_{gh}|-|c_{gh}+1|+|c_{ih}|-|c_{ih}-1+1| =$$

$$= c_{ij}-(c_{ij}-1)-c_{gh}+(c_{gh}+1) = 2.$$

Summarizing all above mentioned we conclude that by r-transfers one can pass from A, B to matrices M, N respectively such that

$$\|M-N\| = \sum_{s,t} |k_{st}-\bar{k}_{st}|.$$

This value is the least among possible ones. ∎

3. Corollaries of the main result

Obviously, the technique of the proof of Theorem 1 can be used, even with some simplifications, to obtain the following analogue of that concerning sets of asymmetric matrices.

Theorem 2. *Let $A\in U_1$, $B\in U_2$ and let U_1, U_2 be compatible. Let $|r_i-r_j|\leqq 1$, $|\bar r_i-\bar r_j|\leqq 1$ ($|s_i-s_j|\leqq 1$, $|\bar s_i-\bar s_j|\leqq 1$ respectively) whenever $x\neq 1$, $z\neq 1$ and $i, j\in I_k$ ($i, j\in J_k$ respectively).*

Then there exist two sequences of r-transfers which transform A, B into matrices M, N respectively such that

$$\|M-N\| = \sum_{s,t} |k_{st}-\bar k_{st}|$$

and this value is the least among possible ones.

Remark 1. The Theorems 1—2 may fail to hold if the conditions $|r_i-r_j|\leqq 1$, $|\bar r_i-\bar r_j|\leqq 1$, $|s_i-s_j|\leqq 1$, $|\bar s_i-\bar s_j|\leqq 1$ are omitied. Thus, let $R=S=\bar R=\bar S=(3, 1, 1, 3)$,

$$K=\bar K=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix},$$

$r=1$, $I_1=J_1=\{1, 2\}$, $I_2=J_2=\{3, 4\}$. Obviously, in this case $U_1=U_2=\{A, B\}$ where

$$A=\begin{bmatrix}1&1&0&1\\0&0&0&1\\1&0&0&0\\1&0&1&1\end{bmatrix},\quad B=\begin{bmatrix}1&0&1&1\\1&0&0&0\\0&0&0&1\\1&1&0&1\end{bmatrix}.$$

But

$$\|A-B\| = 8 > \sum_{s,t} |k_{st}-\bar k_{st}| = 0.$$

Remark 2. The Theorems 1—2 may fail to hold if the condition of compatibility is omitied. Thus, let $S=(2, 1, 2, 1)$, $R=(3, 3)$, $\bar S=(2, 1, 0, 1)$, $\bar R=(2, 2)$, $r=1$, $I_1=\{1\}$, $I_2=\{2\}$, $J_1=\{1, 2\}$, $J_2=\{3, 4\}$,

$$K=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad \bar K=\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}.$$

Obviously, in this case $|U_1|=|U_2|=1$ and

$$A=\begin{bmatrix}1&1&1&0\\1&0&1&1\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&1\\1&1&0&0\end{bmatrix}$$

where $A\in U_1$, $B\in U_2$. But

$$\|A-B\| = 6 > \sum_{s,t} |k_{st}-\bar k_{st}| = 4.$$

Remark 3. The Theorems 1—2 may fail to hold even though partitions of I, J are trivial and the condition of compatibility is omitied. Thus, let

$R=S=(8, 8, 4, 4, 4, 2, 2, 1, 1)$, $\bar{R}=\bar{S}=(4, 4, 4, 4, 0, 0, 0, 0, 0)$, $r=1$. Obviously, in this case $|U_4|=1$, $||B[\{1, 2, 3, 4\}, \{1, 2, 3, 4\}]||=16$, where $B\in U_4$. It is immediately verified that for all $A\in U_3$ $||A[\{1, 2, 3, 4\}, \{1, 2, 3, 4\}]||\leqq 15$, i.e. $B\nleqq A$.

Observe also that $U(R-\bar{R}, S-\bar{S}, 1)\neq\emptyset$. Thus, the last of the above examples disproves conjecture of Brualdi—Anstee [3].

Theorem 3. *If G and H are r-graphs with the same edge degree sequence* (2) *then there is a sequence of r-exchanges which transforms G into a graph isomorphic to H.*

Proof. The sequence (2) uniquely determines vertex degree sequence (1) of r-graphs G, H without isolated vertices. Number vertices of G, H so that identical vertices have the same degrees. Suppose that $i\in I_k$ if and only if $r_i=d_k$. Then adjacency matrices of r-graphs G, H belong to $U(R, K, r|\bigcup_{1}^{x} I_l)$ where $K=[2^{\delta_{ij}}k_{ij}]$. Since a r-exchange of r-graph corresponds uniquely to a symmetric r-transfer of its adjacency matrix the theorem now follows from Theorem 1. ∎

Corollary 1. *The property of "having a fixed edge degree sequence" is s-complete.* ∎

Since a r-switching of r-graph corresponds uniquely to a symmetric r-interchange of its adjacency matrix the following corollaries are immediate consequences of Theorem 1.

Corollary 2 [5]. *If G and H are r-graphs with the same vertex degree sequence then there is a sequence of r-switchings which transforms G into a r-graph isomorphic to H.* ∎

Corollary 3. [13], [14]. *Let sequence* (1) *be 1-graphical, $\bar{R}=(r_1-k_1, ..., r_m-k_m)$, $0\leqq\bar{R}\leqq R$, $|k_i-k_j|\leqq 1$ for all i, j. Then exists a realization of* (1) *containing a $(k_1, ..., k_m)$-factor if and only if $\bar{R}$ is 1-graphical.* ∎

The following result is an immediate consequence of Theorem 2.

Corollary 4. *If matrices $A, B\in U(R, S, r)$ then there is a sequence of r-interchanges which transforms A into B.*

Setting $r=1$, Corollary 4 becomes the well-known Ryser's Theorem [16]. Also Corollary 4 is a special case of Corollary 2.4 [17].

Theorem 4. *Let $U(R, S, r)$, $U(\bar{R}, \bar{S}, r)$ be nonempty and column (row respectively) compatible, $R-\bar{R}\geqq 0$ ($S-\bar{S}\geqq 0$ respectively). Then there is a matrix $A\in U(R, S, r)$ and matrix $B\in U(\bar{R}, \bar{S}, r)$ such that $B\leqq A$.*

Proof. Let $U(R, S, r)$, $U(\bar{R}, \bar{S}, r)$ be column compatible and let A, B belong to these sets respectively, $C=A-B$. Note that in the course of proof of Theorem 1 row compatibility is used in the situation when two entries of opposite signs belonging to the same column of matrix C are considered and partitions of I, J are trivial: for example, $c_{ij}>0$, $c_{gh}<0$, $j=h$. But $r_g-\bar{r}_g\geqq 0$. Hence there exists $s\neq h$ such that $c_{gs}>0$, and in this situation, for entries c_{gh}, c_{gs}, column compatibility is applyed. ∎

Setting $r=1$, Theorem 4 becomes the Anstee—Brualdi's Theorem [3].

References

[1] C. BERGE, *Graph and Hypergraphs*, Dunod, Paris, 1970.
[2] D. BILLINGTON, Connected subgraphs of the graph of multigraphic realizations of a degree sequences, *Lect. Notes Math.* **884** (1981), 125—135.
[3] R. A. BRUALDI, Matrices of zeros and ones with fixed row and column sum vectors, *Lin. Al. Appl.* 33 (1980), 159—231.
[4] M. CAPOBIANCO, S. MAURER, D. MCCARTHY and J. MOLLUZO, A collection of open problems, *Ann. N.-Y. Acad. Sci.* **319** (1979), 565—590.
[5] V. CHANGPHAISAN, Conditions for sequence to be r-graphic, *Discrete Math.* **7** (1974), 31—39.
[6] ZH. A. CHERNYAK, Characterization of self-complimentary degree sequences, *Doklady Acad. Nayk BSSR* **27** (1983), 497—500.
[7] ZH. A. CHERNYAK, Connected realizations of edge degree sequences, *Izvestia Akad. Nauk BSSR. Ser. Fiz.-Mat. Nayk* **3** (1982), 43—47.
[8] ZH. A. CHERNYAK and A. A. CHERNYAK, Edge degree sequences and their realizations, *Doklady Akad. Nauk BSSR* **25** (1981), 594—598.
[9] C. J. COLBOURN, Graph generation. *Dept. of Computer Science, University of Waterloo, Canada, Tech. Report CS—77—37*, 1977.
[10] P. DAS, Unigraphic and Unidigraphic degree sequences through uniquely realizable integer-pair sequences, *Discrete Math.* **45** (1983), 45—59.
[11] R. B. EGGLETON and D. A. HOLTON, Graphic sequences, *Lect. Notes Math.* **748** (1979), 1—10.
[12] D. R. FULKERSON, A. J. HOFFMAN and M. H. MCANDREW, Some properties of graphs with multiple edges, *Canad, J. Math.* **17** (1965), 166—177.
[13] D. J. KLEITMAN and D. L. WANG, Algorithms for constructing graphs, digraphs with given valences and factors, *Discrete Math.* **6** (1973), 79—88.
[14] S. KUNDU, The k-factor conjecture is true, *Discrete Math.* **6** (1973), 367—376.
[15] A. N. PATRINOS and S. L. HAKIMI, Relations between graphs and integerpair sequences, *Discrete Math.* **15** (1976).347—358.
[16] H. J. RYSER, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* **9** (1957), 371—377.
[17] R. P. ANSTEE, The network flows approach for matrices with given row and column sums, *Discrete Math.* **44** (1983), 125—138.

Zh. A. Chernyak

Dept. of Computer Technics,
Radiotechnical Institute,
Minsk, 220013, USSR

A. A. Chernyak

Institute of Problems of Machine Reliability,
Academy of Sciences of BSSR,
Minsk, 2200732, USSR